Problem 33) With reference to Problem 35, where the periodic function $f(x)$ with period $p=1$ is defined over the interval $0 \le x < 1$ as $f(x) = ½ - x$, it is clear that $\phi_1(x) = \pi f(x/2\pi)$. Since $f(x) = \sum_{n=1}^{\infty} \sin(2\pi n x)/(\pi n)$, we will have $\phi_1(x) = \sum_{n=1}^{\infty} \sin(nx)/n$.

Let $\phi_2(x)$ be related to $\int_0^x \phi_1(y)\mathrm{d}y$. In the interval $0 \le x \le \pi$ we will have

$$\int_0^x \phi_1(y)\mathrm{d}y = ½\int_0^x (\pi - y)\mathrm{d}y = \frac{\pi x}{2} - \frac{x^2}{4} = \frac{\pi^2 - (\pi - x)^2}{4}.$$

Similarly, in the interval $-\pi \le x \le 0$, we will have

$$\int_0^x \phi_1(y)\mathrm{d}y = -½\int_0^x (\pi + y)\mathrm{d}y = -\frac{\pi x}{2} - \frac{x^2}{4} = \frac{\pi^2 - (\pi + x)^2}{4}.$$

Next, we integrate the Fourier series representation of $\phi_1(x)$, as follows:

$$\int_0^x \phi_1(y)\mathrm{d}y = \int_0^x \sum_{n=1}^{\infty}[\sin(ny)/n]\,\mathrm{d}y = -\sum_{n=1}^{\infty}[\cos(ny)/n^2]_{y=0}^{x}$$

$$= -\sum_{n=1}^{\infty}[\cos(nx)/n^2] + \sum_{n=1}^{\infty}(1/n^2) = -\sum_{n=1}^{\infty}[\cos(nx)/n^2] + \zeta(2)$$

$$= -\sum_{n=1}^{\infty}[\cos(nx)/n^2] + (\pi^2/6).$$

Riemann's ζ function

Combining the above results, we now find

$$-\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2} + \frac{\pi^2}{6} = \frac{1}{4}\begin{cases}\pi^2 - (\pi + x)^2; & -\pi \le x \le 0\\ \pi^2 - (\pi - x)^2; & 0 \le x \le \pi\end{cases}$$

$$\rightarrow \quad \phi_2(x) = \sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2} = \begin{cases}¼(\pi + x)^2 - (\pi^2/12); & -\pi \le x \le 0\\ ¼(\pi - x)^2 - (\pi^2/12); & 0 \le x \le \pi.\end{cases}$$